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Vertex operators of $G_2^{(1)}$ and $B_l^{(1)}$

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Abstract. The vertex-operator construction is given for the level-one representations of the affine Kac-Moody algebras $G_2^{(1)}$ and $B_l^{(1)}$.

1. Introduction

It is well known to us all that Frenkel and Kac [1, 5, 6] and Segal [2] have constructed the level-one representations of affine Kac-Moody algebras $A_l^{(1)}$, $D_l^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ by means of vertex operators. In 1986, Goddard *et al* [3] also gave the level-one representations of $B_l^{(1)}$, $C_l^{(1)}$, $G_2^{(1)}$ and $F_4^{(1)}$ by vertex operators. On the other hand, Lepowsky and Prime [4] introduced another set of vertex operators in 1984, by which they constructed a level-one representation of $B_l^{(1)}$ as well.

In this paper, we introduce some new vertex operators (see (2.15), (2.16), (3.1) and (5.14), (5.15), (5.18)) which are different from all those in [3, 4]. By using these operators we construct the level-one vertex representations of $G_2^{(1)}$ and $B_l^{(1)}$, prove them to be the direct sums of irreducible subrepresentations, and also give their representation spaces.

2. Some lemmas

Let H denote the Cartan subalgebra of the finite-dimensional simple Lie algebra B_l over C, and let H^* be the dual space of H_R. Then there exist the normal orthogonal basis $e_1, \ldots, e_l$ and the inner product (a, b), such that the short root system $\Delta_S = \{\pm e_i, 1 \leqslant i \leqslant l\}$ and the long root system $\Delta_L = \{\pm e_i + e_j, \pm e_i - e_j, 1 \leqslant i < j \leqslant l\}$. We have $(\Delta_L + \Delta_L) \cap \Delta \subset \Delta_L$, $(\Delta_L + \Delta_S) \cap \Delta \subset \Delta_S$, $(\Delta_S + \Delta_S) \cap \Delta \subset \Delta_L$.

The simple root system $\Pi = \{e_1 - e_2, \ldots, e_{l-1} - e_l, e_l\}$.

Let $a_1 = e_1 - e_2, \ldots, a_{l-1} = e_{l-1} - e_l, a_l = e_l$, so the root lattice

$$L = \left\{ \sum_{j=1}^{l} m_j a_j \,\middle|\, m_1, \ldots, m_l \in \mathbb{Z} \right\} = \left\{ \sum_{j=1}^{l} m_j e_j \,\middle|\, m_1, \ldots, m_l \in \mathbb{Z} \right\}. \tag{2.1}$$

Lemma 2.1. Suppose that the mapping p: $\Delta_S \to L$ satisfies the conditions

(i) $p(a) = p(a+b), \forall a \in \Delta_S, b \in \Delta_L, a+b \in \Delta$;

(ii) $(a, -a) + (p(a), p(-a))$ is an even number, $\forall a \in \Delta_S$;

(iii) $(a, b) + (p(a), p(b))$ is an odd number, $\forall a, b \in \Delta_S, a+b \in \Delta$;

(iv) $(a, b) \geqslant |(p(a), p(b))|, \forall a, b \in \Delta_S, a+b \bar{\in} \{0\} \cup \Delta$,

then $p(a) = e$, where e is any fixed element in Δ_S.

Proof. For short root e_l, we have $(\pm e_i - e_l) + e_l = \pm e_i$, $i < l$ and $(-e_1 - e_l) + e_1 = -e_l$. By (i) we proved that $p(a) = e, \forall a \in \Delta_S$, where $e \in L$. But $2 = (a+b, a+b) = 2 + 2(a, b)$, i.e. $(a, b) = 0, \forall a, b \in \Delta_S, a+b \in \Delta_L$. By (iii), we know that (e, e) is a positive odd integer, hence (ii) holds. Finally, let $a = \pm e_i$, $b = \pm e_j$, if $a + b \neq 0, a + b \bar{\in} \Delta$, then $a = b = \pm e_i$, so $(e, e) = 1$ by (iv), i.e. e is a short root. □

For convenience, we only discuss the case when $e = e_l$ in the following.

Lemma 2.2. If the mapping $\varepsilon : L \times L \to \{\pm 1\}$ defined by

$$\varepsilon\left(\sum_{i=1}^{l} m_i e_i, \sum_{j=1}^{l} n_j e_j\right) = \prod_{i,j=1}^{l} \varepsilon(e_i, e_j)^{m_i m_j} \tag{2.2}$$

where

$$\varepsilon(e_i, e_j) = 1 \quad 1 \leqslant i \leqslant j \leqslant l \qquad \varepsilon(e_j, e_i) = -\varepsilon(e_i, e_j) \quad i < j \tag{2.3}$$

then we have

(i) $\varepsilon(a, b)\varepsilon(a+b, c) = \varepsilon(b, c)\varepsilon(a, b+c), \forall a, b, c \in L$,
(ii) $\varepsilon(a, -a) = -1, \forall a \in \Delta_L$,
(iii) $\varepsilon(a, -a) = 1, \forall a \in \Delta_S$,
(iv) $\varepsilon(a, b)\varepsilon(b, a) = -(-1)^{(a,b)}, \forall a, b \in \Delta_S, a + b \neq 0$,
(v) $\varepsilon(a, b)\varepsilon(b, a) = (-1)^{(a,b)}, \forall a \in \Delta_L, b \in \Delta, a + b \neq 0$.

Proof. By (2.2), both (ii) and (iii) are true, and

$$\varepsilon(a+b, c) = \varepsilon(a, c)\varepsilon(b, c) \qquad \varepsilon(a, b+c) = \varepsilon(a, b)\varepsilon(a, c)$$

so (i) holds. Let $a = \Sigma_{i=1}^{l} m_i e_i$, $b = \Sigma_{i=1}^{l} n_i e_i$, then

$$\varepsilon(a, b)\varepsilon(b, a) = (-1)^{(a,b)+(\Sigma m_i)(\Sigma n_i)}.$$

If $a \in \Delta_L$, then $\Sigma m_i \equiv 0$, (mod 2); if $b \in \Delta_S$ then $\Sigma m_i \equiv 1$, (mod 2), so (iv) and (v) are true. □

Let $a_1^\vee, \ldots, a_l^\vee$, and $e_a, \forall a \in \Delta$ be the Chevalley basis of B_l, and the multiplication table is

$$[a_i^\vee, a_j^\vee] = 0 \qquad [a_i^\vee, e_a] = a(a_i^\vee)e_a \qquad [e_a, e_{-a}] = -\gamma^{-1}(a) \qquad \forall a \in \Delta_L$$

$$[e_a, e_{-a}] = 2\gamma^{-1}(a) \quad a \in \Delta_S \qquad [e_a, e_b] = \varepsilon(a, b)e_{a+b} \quad \forall a \in \Delta_L \quad b, a+b \in \Delta$$

$$[e_a, e_b] = 2\varepsilon(a, b)e_{a+b} \quad \forall a, b \in \Delta_S \quad a + b \in \Delta$$

$$[e_a, e_b] = 0 \quad \forall a, b \in \Delta \quad 0 \neq a + b \bar{\in} \Delta$$

where $\varepsilon(a, b)$ is defined in lemma 2.2, so we have $\varepsilon(a, b) = -\varepsilon(b, a)$, and $\gamma : \mathrm{H} \to \mathrm{H}^*$, such that

$$\gamma(a_i^\vee) = a_i \quad 1 \leqslant i < l \qquad \gamma(a_l^\vee) = 2a_l \tag{2.4}$$

then

$$(a, b) = b(\gamma^{-1}(a)) \quad \forall a, b \in \mathrm{H}^* \qquad (x, y) = \gamma(x)(y) \quad \forall x, y \in \mathrm{H}. \tag{2.5}$$

Since affine Lie algebra $B_l^{(1)}$ has a basis c, d, $t^k\otimes a_i^\vee$, $1\le i\le l$, $t^k\otimes e_a, \forall a\in\Delta$, where $k\in Z$, and

$$[c,d]=0 \qquad [c,t^k\otimes x]=0 \qquad [d,t^k\otimes x]=k(t^k\otimes x) \qquad \forall x\in B_l$$
$$[t^j\otimes\gamma^{-1}(a),t^k\otimes\gamma^{-1}(b)]=j\delta_{j,-k}(a,b)c \qquad \forall a,b\in H^*$$
$$[t^j\otimes\gamma^{-1}(a),t^k\otimes e_b]=(a,b)t^{j+k}\otimes e^b \qquad a\in H^* \quad b\in\Delta$$
$$[t^j\otimes e_a,t^k\otimes e_{-a}]=\varepsilon(a,-a)(t^{j+k}\otimes\gamma^{-1}(a)+j\delta_{j,-k}c) \qquad \forall a\in\Delta_L$$
$$[t^j\otimes e_a,t^k\otimes e_{-a}]=2\varepsilon(a,-a)(t^{j+k}\otimes\gamma^{-1}(a)+j\delta_{j,-k}c) \qquad \forall a\in\Delta_S$$
$$[t^j\otimes e_a,t^k\otimes e_b]=\varepsilon(a,b)t^{j+k}\otimes e_{a+b} \qquad \forall a\in\Delta_L \quad b,a+b\in\Delta$$
$$[t^j\otimes e_a,t^k\otimes e_b]=2\varepsilon(a,b)t^{j+k}\otimes e_{a+b} \qquad \forall a,b\in\Delta_S \quad b,a+b\in\Delta$$
$$[t^j\otimes e_a,t^k\otimes e_b]=0 \quad \forall a,b\in\Delta \quad a+b\bar{\in}\Delta\cup\{0\}$$

where $j,k\in\mathbb{Z}$ and δ_{pq} is the Kronecker symbol.

Introducing e^c for all $c\in L$, and

$$e^ae^b=e^be^a=e^{a+b} \qquad \forall a,b\in L, \tag{2.6}$$

then

$$e^L=\{e^c|\forall c\in L\} \tag{2.7}$$

is an Abelian group, its group algebra over C denoted by $C\{L\}$.

Denote

$$h_i(n)=t^n\otimes\gamma^{-1}(e_i) \qquad 1\le i<l \quad n\in\mathbb{Z} \tag{2.8}$$
$$h_l(n/2)=t^{n/2}\otimes\gamma^{-1}(e_l) \qquad n\in\mathbb{Z} \tag{2.9}$$

and S^- is the linear space over C with the basis

$$\{1,h_1(n),\ldots,h_{l-1}(n),h_l(n/2) \qquad \forall n\in\mathbb{Z}^--\{0\}\}. \tag{2.10}$$

Let $S(S^-)$ be the symmetric tensor algebra generated by S^-, its product denoted by $\vee$, hence $S(S^-)$ is a commutative associate algebra. Let

$$V=S(S^-)\otimes c\{L\} \tag{2.11}$$

then V has a basis which we shall always denote in this paper by $v\otimes e^c$, $c\in L$, where v is among the following:

$$v=1 \qquad \text{or} \qquad v=h_{i_1}(n_1)\vee\ldots\vee h_{i_t}(n_t) \tag{2.12}$$

here $t=1,2,\ldots,1\le i_1\le\ldots\le i_t\le l$. When $i_j\ne l$, then $n_j\in\mathbb{Z}^--\{0\}$, when $i_j=l$, then $n_j\in\frac{1}{2}\mathbb{Z}^--\{0\}$.

The formal linear combination of infinite elements of the basis in V generates the complete space $\tilde{V}$ of V. We give some operators in the following:

(i) let c_0 be the identity operator on $\tilde{V}$;

(ii) $d_0(v\otimes e^c)=\deg(v\otimes e^c)(v\otimes e^c)$ where $\deg(1)=0$, and $\deg(v\otimes e^c)=\Sigma_{j=1}^l n_j-\frac{1}{2}(c,c)$;

(iii) $\varepsilon_a(v\otimes e^c)=\varepsilon(a,c)(v\otimes e^c)$;

(iv) $H_i(0)(v\otimes e^c)=(c,e_i)(v\otimes e^c)$

$$H_i(n)(v\otimes e^c)=(h_i(n)\vee v)\otimes e^c \qquad n\in\begin{cases}\mathbb{Z}^--\{0\} & i<l\\ \frac{1}{2}\mathbb{Z}^--\{0\} & i=l\end{cases}$$
$$H_i(n)(v\otimes e^c)=(\partial_{h_i(n)}v)\otimes e^c \qquad n\in\begin{cases}\mathbb{Z}^+-\{0\} & i<l\\ \frac{1}{2}\mathbb{Z}^+-\{0\} & i=l\end{cases}$$

where

$$\partial_{h_i(n)}1 = 0 \qquad \partial_{h_i(n)}h_j(m) = n\delta_{n,-m}\delta_{ij}$$

$$\begin{aligned}\partial_{h_i(n)}(h_{i_1}(n_1) \vee \dots \vee h_{i_t}(n_t)) \\ &= \sum_{j=1}^{t} (\partial_{h_i(n)}h_{i_j}(n_j))h_{i_1}(n_1) \vee \dots \vee h_{i_{j-1}}(n_{j-1}) \\ &\quad \vee \widehat{h_{i_j}(n_j)} \vee h_{i_{j+1}}(n_{j+1}) \vee \dots \vee h_{i_t}(n_t)\end{aligned}$$

where $\widehat{h_{i_j}(n_j)}$ means $h_{i_j}(n_j)$ does not appear in that term.

By direct computation, we have the following results.

Lemma 2.3. $[H_i(m), H_j(m)] = n\delta_{n,-m}(e_i, e_j)c_0$, where n and m are chosen as in (4).

Lemma 2.4. Let $a = \Sigma_{i=1}^{l} n_i e_i$, $b = \Sigma_{j=1}^{l} m_j e_j$, $\forall a, b \in L$. If

$$a(n) = \sum_{i=1}^{l} n_i H_i(n) \qquad b(m) = \sum_{j=1}^{l} m_j H_j(m)$$

then

$$[a(n), b(m)] = n\delta_{n,-m}(a, b)c_0. \tag{2.13}$$

(v) Let z be a complex variable, denote

$$\tilde{X}(a, z)(v \otimes e^c) = z^{(a,a)+2(a,c)}\varepsilon(a, c)(v \otimes e^{a+c}) \qquad \forall a \in \Delta \tag{2.14}$$

$$E^{\pm}(a, z) = \exp\left(\sum_{n=1}^{\infty} \frac{\pm 1}{n} z^{\mp 2n} a(\mp n)\right) \qquad \forall a \in \Delta \tag{2.15}$$

$$F^{\pm}(e_l, z) = \exp\left(\sum_{n=1}^{\infty} \frac{\pm 2}{2n+1} z^{\pm(2n+1)} e_l\left(\mp\frac{2n+1}{2}\right)\right) \tag{2.16}$$

then we have the following lemma.

Lemma 2.5. Suppose that z and w are the complex variables, then we have

$$[E^{\pm}(a, z), E^{\pm}(b, w)] = [E^{\pm}(a, z), F^{+}(e_l, w)] = [E^{\pm}(a, z), F^{-}(e_l, w)] = 0$$

$$[F^{\pm}(e_l, z), F^{\pm}(e_l, w)] = 0$$

$$E^{\pm}(a, z)E^{\pm}(b, z) = E^{\pm}(a+b, z) \qquad F^{\pm}(e_l, z)F^{\pm}(e_l, -z) = c_0$$

$$E^{-}(a, z)E^{+}(b, w)E^{-}(a, z)^{-1}E^{+}(b, w)^{-1} = z^{-2(a,b)}(z-w)^{(a,b)}(z+w)^{(a,b)}c_0 \qquad |w| < |z| \tag{2.17}$$

$$F^{-}(e_l, z)F^{+}(e_l, w)F^{-}(e_l, z)^{-1}F^{+}(e_l, w)^{-1} = (z-w)^{(a,b)}(z+w)^{-(a,b)}c_0 \qquad |w| < |z|. \tag{2.18}$$

Proof. By (1.5), we have

$$a(n)^p b(-n)^q = \sum_{i=0}^{\min(p,q)} \frac{p!q!}{i!(p-i)!(q-i)!} n^i (a, b)^i b(-n)^{p-i} a(n)^{q-i}$$

therefore we can prove this lemma by the direct computation. □

3. Vertex representation of $\mathbf{B}_l^{(1)}$

Definition. Let z be a complex variable, and

$$X(a,z)=\begin{cases} E^{+}(a,z)E^{-}(a,z)\tilde{X}(a,z)\varepsilon_a, & \forall a\in\Delta_{\mathrm{L}} \\ \sqrt{-1}F^{+}(e_l,z)F^{-}(e_l,z)E^{+}(a,z)E^{-}(a,z)\tilde{X}(a,z)\varepsilon_a, & \forall a\in\Delta_{\mathrm{S}}. \end{cases} \tag{3.1}$$

The Laurent series of the operator $X(a, z)$ is denoted by

$$X(a,z)=\sum_{n=-\infty}^{\infty}(X_{n/2}(a))z^{-n}.$$

The $X_n(a), \forall n\in Z$ are operators on V, called the vertex operators of a.

Theorem 1. Affine Lie algebra $\mathrm{B}_l^{(1)}$ has a linear representation (V, ρ) such that

$$\rho(c)=c_0 \qquad \rho(d)=d_0 \qquad \rho(t^n\otimes\gamma^{-1}(e_i))=H_i(n) \qquad 1\leqslant i\leqslant l \tag{3.2}$$

$$\rho(t^n\otimes e_a)=X_n(a) \qquad \forall a\in\Delta. \tag{3.3}$$

Definition. Linear representation (V, ρ) is called vertex representation of $\mathrm{B}_l^{(1)}$.

Proof of theorem 1. By the multiplication table of $\mathrm{B}_l^{(1)}$, we only prove

$$[X_j(a),X_k(-a)]=\frac{2}{(a,a)}\varepsilon(a,-a)(a(j+k)+j\delta_{j,-k}c_0) \qquad a\in\Delta \tag{3.4}$$

$$[X_j(a),X_k(b)]=(2+(a,b))\varepsilon(a,b)X_{j+k}(a,b) \qquad a,b,a+b\in\Delta \tag{3.5}$$

$$[X_j(a),X_k(b)]=0 \qquad a,b\in\Delta,\ a+b\,\bar{\in}\,\Delta\cup\{0\}. \tag{3.6}$$

Denote

$$\begin{aligned} Y&(a,b;z,z_0)\\ &=(\sqrt{-1})^{(p,p)+(q,q)}F^{+}(p,z)F^{+}(q,z_0)F^{-}(p,z)F^{-}(q,z_0)\\ &\quad\times E^{+}(a,z)E^{+}(b,z_0)E^{-}(a,z)E^{-}(b,z_0)\tilde{X}(a+b,z_0)\varepsilon_{a+b} \end{aligned} \tag{3.7}$$

where (i) $a\in\Delta_{\mathrm{L}}, p=0$, (ii) $a\in\Delta_{\mathrm{S}}, p=e_l$, (iii) $b\in\Delta_{\mathrm{L}}, q=0$, (iv) $b\in\Delta_{\mathrm{S}}, q=e_l$, and let

$$f_r(a,b;z,z_0)=z^{2n-1}z_0^{-2(a,b)}(zz_0^{-1})^{(a,a)+2(a,r)}(z-z_0)^{(a,b)+(p,q)}(z+z_0)^{(a,b)-(p,q)}Y(a,b;z,z_0) \tag{3.8}$$

then we have

$$z^{2n-1}X(a,z)X(b,z_0)(v\otimes e^r)=\varepsilon(a,b)f_r(a,b;z,z_0)(v\otimes e^r) \qquad |z|<|z_0|$$

$$z^{2n-1}X(b,z_0)X(a,z)(v\otimes e^r)=\varepsilon(b,a)(-1)^{(a,b)+(p,q)}f_r(a,b;z,z_0)(v\otimes e^r) \qquad |z_0|<|z|.$$

On the other hand, we have

$$\varepsilon(a,b)=\varepsilon(b,a)(-1)^{(a,b)+(p,q)} \qquad \forall a,b\in\Delta$$

by lemma 2.2, therefore

$$\begin{aligned} [X_n&(a),X(b,z_0)](v\otimes e^r)\\ &=\mathrm{Res}_{z=0}\,z^{2n-1}X_n(a,z)X(b,z_0)(v\otimes e^r)\\ &\quad-\mathrm{Res}_{z=0}\,z^{2n-1}X(b,z_0)X_n(a,z)(v\otimes e^r) \end{aligned}$$

where $n \in \mathbb{Z}$, $a, b \in \Delta$, and z, z_0 are two independent complex variables. It is well known that

$$[X_n(a), X(b, z_0)](v \otimes e^r) = \frac{\varepsilon(a, b)}{2\pi\sqrt{-1}} \int_D f_r(a, b; z, z_0)(v \otimes e^r)\, dz \tag{3.9}$$

where $D = \{z \in C \mid 0 < r_0 < |z| < r_1\}$ such that $r_0 < |z_0| < r_1$.

If $a \in \Delta_L$, then $p = q = 0$, $(r, a) \in Z$, $\forall r \in L$, and

$$f_r(a, -a; z, z_0) = z^{2n-1+2(a,r)} z_0^{2-(a,r)} (z - z_0)^{-2} (z + z_0)^{-2} Y(z, -z; z, z_0)$$

then we have

$$\begin{aligned}
&[X_n(a), X(-a, z_0)](v \otimes e^r) \\
&\quad = \varepsilon(a, -a) \frac{d}{dz} (z - z_0)^2 f_r|_{z = z_0} (v \otimes e^r) \\
&\qquad + \varepsilon(a, -a) \frac{d}{dz} (z + z_0)^2 f_r|_{z = -z_0} (v \otimes e^r) \\
&\quad = \varepsilon(a, -a)(n + (a, r)) z_0^{2n} (v \otimes e^r) \\
&\qquad + \varepsilon(a, -a) z_0^{2n} \left(\sum_{m=-\infty}^{\infty} z_0^{2m} a(-m) - a(0) \right) (v \otimes e^r) \\
&\quad = \varepsilon(a, -a) \left(\sum_{m=-\infty}^{\infty} z_0^{2(m+n)} a(-m) + n z_0^{2n} \right) (v \otimes e^r).
\end{aligned}$$

So (3.4) holds when $a \in \Delta_L$.

If $a \in \Delta_S$, then $p = q = e_l$, $(r, a) \in Z$, $\forall r \in L$, and

$$f_r(a, -a; z, z_0) = z^{2n+2(a,r)} z_0^{1-2(a,r)} (z + z_0)^{-2} Y(a, a; z, z_0).$$

then we have

$$\begin{aligned}
[X_n(a), X(-a, z_0)](v \otimes e^r) &= \varepsilon(a, -a) \frac{d}{dz} (z + z_0)^2 f_r|_{z = z_0} (v \otimes e^r) \\
&= 2\varepsilon(a, -a) \left[n z_0^{2n} - z_0^{2n} \sum_{m=-\infty}^{\infty} z_0^{2m+1} e_l \left(-\frac{2m+1}{2} \right) \right. \\
&\qquad \left. + z_0^{2n} \sum_{m=-\infty}^{\infty} z_0^{2m} a(-m) \right] (v \otimes e^r)
\end{aligned}$$

therefore (3.4) holds when $a \in \Delta_S$.

When $a, b \in \Delta$ and $a + b \neq 0$, $a + b \bar{\in} \Delta$, then f_r is not a singularity by lemma 2.1, therefore the integral is equal to zero, hence (3.6) holds.

If $a, b \in \Delta_S$, and $a + b \in \Delta_L$, $p = q = e_l$, in this case

$$f_r(a, b; z, z_0) = z^{2n+2(a,r)} z_0^{-1-2(a,r)} (z - z_0)(z + z_0)^{-1} Y(a, b; z, z_0).$$

then we have

$$\begin{aligned}
[X_n(a), X(b, z_0)](v \otimes e^r) &= \varepsilon(a, b)(z + z_0) f_r|_{z = z_0} (v \otimes e^r) \\
&= 2\varepsilon(a, b) z_0^{2n} X(a + b, z_0).
\end{aligned}$$

So (3.5) holds when $a, b \in \Delta_S$ and $a + b \in \Delta_L$.

Finally, if $a \in \Delta_L$, $b \in \Delta_S$, and $a+b \in \Delta_S$, then $p=0$ and $q=e_l$. In this case

$$f_r(a, b; z, z_0) = z^{2n+1+2(a,r)} z_0^{-2(a,r)} (z-z_0)^{-1}(z+z_0)^{-1} Y(a, b; z, z_0)$$

then we have

$$\begin{aligned}[X_n(a), X(b, z_0)](v\otimes e^r) \\ &= \varepsilon(a, b)(z-z_0) f_r|_{z=z_0}(v\otimes e^r) + \varepsilon(a, b)(z+z_0) f_r|_{z=-z_0}(v\otimes e^r) \\ &= \varepsilon(a, b) z_0^{2n} X(a+b, z_0)(v\otimes e^r).\end{aligned}$$

So (3.7) holds when $a \in \Delta_L$, $b \in \Delta_S$ and $a+b \in \Delta_S$. This proves the theorem. □

Similarly, we can put $p(a) = e_i \forall a \in \Delta_S$ where i is any one of $1, 2, \ldots, l, -1, -2, \ldots, -l$, and we can make the vertex operators and the vertex representations as before. So there are $2l$ vertex representations of the affine Lie algebra $B_l^{(1)}$.

4. The weight system of the vertex representations of $B_l^{(1)}$

Let H_1 be the Cartan subalgebra of $B_l^{(1)}$, H_1^* is the dual space of H_1, then there exists a basis of H_1 as follows:

$$a_{-1}^\vee = d \qquad a_0^\vee = c - a_1^\vee - 2a_2^\vee - \ldots - 2a_{l-1}^\vee - a_l^\vee \qquad a_1^\vee, \ldots, a_l^\vee \tag{4.1}$$

and a basis of H_1^* in the following way:

$$a_{-1}, a_0, a_1, \ldots, a_l \tag{4.2}$$

such that $a_1, \ldots, a_l$ form the simple root system. Since there exists a non-degenerate symmetric bilinear function such that $e_1, \ldots, e_l$ form part of its normal orthogonal basis, where

$$a_1 = e_1 - e_2 \qquad \ldots \qquad a_{l-1} = e_{l-1} - e_l \qquad e_l = e_l \tag{4.3}$$

we therefore have the orthogonal basis

$$\begin{aligned}&\tilde{e}_{-1} = \tfrac{1}{2}a_{-1} + a_0 \qquad \tilde{e}_0 = -\tfrac{3}{2}a_{-1} + a_0 \qquad \tilde{e}_1 = a_{-1} + e_1 \\ &\tilde{e}_2 = a_{-1} + e_2 \qquad \tilde{e}_j = e_j \qquad 3 \leq j \leq l.\end{aligned} \tag{4.4}$$

in H_l^*, and $(\tilde{e}_i, \tilde{e}_i) = 1$, $i \neq 0$, $(\tilde{e}_0, \tilde{e}_0) = -1$.

Now

$$\begin{aligned}&a_{-1} = \tilde{e}_{-1} - \tilde{e}_0 \qquad a_0 = \tfrac{3}{2}\tilde{e}_{-1} - \tfrac{1}{2}\tilde{e}_0 \qquad a_1 = \tilde{e}_1 - \tilde{e}_2 = e_1 - e_2 \\ &a_2 = -\tilde{e}_{-1} + \tilde{e}_0 + \tilde{e}_2 - \tilde{e}_3 = e_2 - e_3 \\ &a_3 = \tilde{e}_3 - \tilde{e}_4 = e_3 - e_4 \qquad \ldots \qquad a_{l-1} = \tilde{e}_{l-1} - \tilde{e}_l = e_{l-1} - e_l \qquad a_l = \tilde{e}_l = e_l.\end{aligned} \tag{4.5}$$

Suppose that $\gamma: H_1 \to H_l^*$, $\gamma(a_i^\vee) = a_i$, $-1 \leq i < l$, $\gamma(a_l^\vee) = 2a_l$, then we have

$$a(x) = (a, \gamma(x)) \quad (a, b) = b(\gamma^{-1}(a)) \quad (x, y) = \gamma(x)(y) \qquad \forall a, b \in H_1^* \quad x, y \in H_1. \tag{4.6}$$

The imaginary root system of $B_l^{(1)}$ is $\Delta_{im} = \{k\delta | k \in Z - \{0\}\}$, where

$$\delta = \gamma(c) = a_0 + a_1 + 2a_2 + \ldots + 2a_l = -\tfrac{1}{2}\tilde{e}_{-1} + \tfrac{3}{2}\tilde{e}_0 + \tilde{e}_1 + \tilde{e}_2 = \tfrac{3}{2}\tilde{e}_{-1} - \tfrac{1}{2}\tilde{e}_0 + e_1 + e_2, \tag{4.7}$$

and

$$(\delta, a_{-1}) = 1 \qquad (\delta, a_0) = 0 \qquad (\delta, L) = 0. \tag{4.8}$$

We know that $\rho(H_1)$ has a basis $d_0, c_0, H_i(0), 1 \leq i \leq l$, so the root subspace decomposition is

$$V = \sum_{\lambda \in P(V)} V_\lambda \tag{4.9}$$

performed by the operator set $\rho(H_1)$, where $P(V) \subset H_1^*$ is the weight system of the vertex representation (V, ρ) of $B_l^{(1)}$.

Lemma 4.1. The weight space V_λ has the basis $v \otimes e^r$, where $r \in L$, and $v = 1$ or $v = h_{i_1}(n_1) \vee \ldots \vee h_{i_t}(n_t)$, and

$$\lambda = a_{-1} + r + \deg(v \otimes e^r)\delta \tag{4.10}$$

where $\deg(v)$ and r are uniquely determined by λ.

Proof. Now that

$$c_0(v \otimes e^r) = v \otimes e^r \qquad d_0(v \otimes e^r) = (\deg(v \otimes e^r))(v \otimes e^r)$$
$$H_i(0)(v \otimes e^r) = (r, e_i)(v \otimes e^r)$$

so $v \otimes e^r$ is a common eigenvector of $\rho(H_1)$, it belongs to weight space V_λ, where $\lambda \in H_1^*$,

$$\lambda(d) = \deg(v \otimes e^r) \quad \lambda(c) = 1 \quad \lambda(\gamma^{-1}(e_i)) = (r, e_i) \qquad 1 \leq i \leq l.$$

Let $\lambda - r = xa_{-1} + y\delta + \Sigma_{j=1}^{l} x_j e_j$, we know $\lambda = xa_{-1} + r + y\delta$ by $(a_{-1}, e_i) = 0, 1 \leq i \leq l$. But $\gamma(c) = \delta$, so

$$1 = \lambda(c) = (\lambda, \gamma(c)) = (\lambda, \delta) = x$$
$$\deg(v \otimes e^r) = \lambda(d) = (\lambda, \gamma(d)) = (\lambda, a_{-1}) = y$$

i.e. $\lambda = a_{-1} + r + (\deg(v \otimes e^r))\delta$. On the other hand, if $v \otimes e^r, \tilde{v} \otimes e^{\tilde{r}} \in V_\lambda$, then $r + \deg(v \otimes e^r)\delta = \tilde{r} + \deg(\tilde{v} \otimes e^{\tilde{r}})\delta$, therefore $\deg(v \otimes e^r) = \deg(\tilde{v} \otimes e^{\tilde{r}})$, so we prove that $\tilde{r} = r$, hence $\deg(v) = \deg(\tilde{v})$. □

Lemma 4.2. For any $\lambda \in P(V)$, we have $\lambda \leq a_{-1}$ or $\lambda \leq a_{-1} + e_1 - \frac{1}{2}\delta$.

Proof. Suppose $\lambda = a_{-1} + r + \deg(v \otimes e^r)\delta = a_{-1} + r + (\deg v - \frac{1}{2}(r, r))\delta$, where $r = \Sigma_{i=1}^{l} k_i a_i \in L$, $k_i \in \mathbb{Z}$, then we have $\frac{1}{2}(r, r) = \frac{1}{2}[k_1^2 + (k_1 - k_2)^2 + \ldots + (k_{l-1} - k_l)^2]$, therefore $\deg(v \otimes e^r) = \deg v - \frac{1}{2}[k_1^2 + (k_1 - k_2)^2 + \ldots + (k_{l-1} - k_l)^2] \in \mathbb{Z}^-$ or $\mathbb{Z}^- - \frac{1}{2}$

$$\begin{aligned}\lambda &= a_{-1} + r + \deg(v \otimes e^r)\delta \\ &= a_{-1} + \sum_{i=1}^{l} k_i a_i + \left(\deg v - \frac{1}{2}\sum_{i=1}^{l}(k_{i-1} - k_i)^2\right)\left(a_0 + a_1 + 2\sum_{j=2}^{l} a_j\right) \\ &= a_{-1} + \sum_{i=0}^{l} m_i a_i\end{aligned}$$

where $k_0 = 0$. If $\deg(v \otimes e^r) \in \mathbb{Z}^-$, then $m_0, \ldots, m_l \in \mathbb{Z}^-$ by direct computation, therefore $\lambda \leq a_{-1}$. Now

$$\lambda - (e_1 - \tfrac{1}{2}\delta) = \lambda + \tfrac{1}{2}(a_0 - a_1) = a_{-1} + \sum_{i=0}^{l} n_i a_i$$

if $\deg(v \otimes e^r) \in \mathbb{Z}^- - \frac{1}{2}$, then $n_0, \ldots, n_l$ are not positive numbers by direct computation, so $\lambda \leq a_{-1} + e_1 - \frac{1}{2}\delta$. □

Lemma 4.3. (V, ρ) is an integrable representation.

Proof. We know that the generators are $t\otimes e_{-\theta}$, $t^{-1}\otimes e_\theta$, $1\otimes e_{a_i}$, $1\otimes e_{-a_i}$, $1\leq i\leq l$, where $\theta=\delta-a_0=e_1+e_2\in\Delta_L$,

$$\rho(t\otimes e_{-\theta})=X_1(-\theta)\qquad \rho(1\otimes e_{a_i})=X_0(a_i)\qquad 1\leq i\leq l$$

$$\rho(t^{-1}\otimes e_\theta)=X_{-1}(\theta)\qquad \rho(1\otimes e_{-a_i})=X_0(-a_i) 1\leq i\leq l.$$

By the definition of the integrable representation, we need to prove that for any $x\in V$, there exists a positive integer N such that

$$X_1(-\theta)^N x=0\qquad X_0(a_i)^N x=0\qquad 1\leq i\leq l \tag{4.11}$$

$$X_{-1}(\theta)^N x=0\qquad X_0(-a_i)^N x=0\qquad 1\leq i\leq l. \tag{4.12}$$

We may assume that $x=v_0\otimes e^r$, where $r\in L$. Let

$$X'(-a_l, z)=F^+(a_l, z)F^-(a_l, z)E^+(a_l, z)E^-(a_l, z)=\sum_{m=-\infty}^{\infty} z^m X'_{-m}(-a_l)$$

then

$$X(-a_l, z)(v_0\otimes e^r)=X'(-a_l, z)v_0\otimes z^{1-2(a_l,r)}\varepsilon(-a_l, r)\, e^{r-a_l}$$

so we have

$$X_0(-a_l)(v_0\otimes e^r)=\varepsilon(-a_l, r)X'_{1-2(a_l,r)}(-a_l)v_0\otimes e^{r-a_l}$$

where $v_1=\varepsilon(-a_l, r)X'_{1-2(a_l,r)}(-a_l)v_0$. If $v_1\neq 0$, then

$$\deg v_1=\deg v_0+1-2(a_l, r).$$

Let

$$X_0(-a_l)^k(v_0\otimes e^r)=v_k\otimes e^{r-ka_l}\qquad k=1,2,\ldots$$

if $v_1,\ldots,v_k\neq 0$, then we have

$$\deg v_k=\deg v_{k-1}+1-2(a_l, r-(k-1)a_l)$$
$$=\deg v_{k-1}-2(a_l, r)+2k-1$$

thus

$$\deg v_k=\deg v_0-2k(a_l, r)+k^2$$

therefore we have proven that there exists a positive integer N such that $v_N=0$, hence $X_0(-a_l)^N(v_0\otimes e^r)=0$.

Similarly, we can prove that (4.11) and (4.12) are also true. □

Lemma 4.4. $P_+\cap P(V)=\{a_{-1}+\deg(v)\delta,\ a_{-1}+e_1+(\deg(v)-\frac{1}{2})\delta\}$.

Proof. For any $\lambda\in P(V)$, then $\lambda=a_{-1}+r+\deg(v\otimes e^r)\delta$ by lemma 4.1. But $\lambda\in P_+$, then $\lambda(a_i^\vee)\in Z^+, 0\leq i\leq l$, i.e. $(\lambda, a_i)\in Z^+, 0\leq i<l$, and $2(\lambda, a_l)\in Z^+$. Suppose $r=\sum_{j=1}^{l} m_je_j$, where $m_1,\ldots,m_l\in\mathbb{Z}$. So we have $m_1\geq\ldots\geq m_l\geq 0$ and $(\lambda, a_0)=1+(r, a_0)=1-m_1-m_2\in\mathbb{Z}^+$, therefore $r=0$ or $r=e_1$. □

Lemma 4.5. Only $1\otimes e^0$ and $1\otimes e^{e_1}$ are the highest-weight vectors in the representation space V, $1\otimes e^0$ has the highest weight a_{-1} and $1\otimes e^{e_1}$ has the highest weight $a_{-1}+e_1-\frac{1}{2}\delta$.

Proof. Let $v=1$ or $v=v_1\vee\ldots\vee v_l\vee\tilde{v}_l$, where $v_i=h_i(n_{i1})\vee\ldots\vee h_i(n_{is_i})$, $1\leqslant i\leqslant l$, $n_{ij}\in\mathbb{Z}^- -\{0\}$ and $\tilde{v}_l=h_l(-2n_l+1/2)\vee\ldots\vee h_l(-2n_s+1/2)$, $n_1,\ldots,n_s\in\mathbb{Z}^+$. Since

$$\tilde{X}(a,z)\varepsilon_a(v\otimes e^r)=z^{(a,a)+2(a,r)}\varepsilon(a,r)(v\otimes e^{r+a}).$$

When $a=a_i\in\Delta_L$, $1\leqslant i\leqslant l-1$, then

$$X(a_i,z)(v\otimes e^r)=E^+(a_i,z)E^-(a_i,z)v\otimes z^{2+2(a_i,r)}\varepsilon(a_i,r)\,e^{a_i+r}.$$

Denote

$$\begin{aligned}\psi&=E^+(a_i,z)E^-(a_i,z)(v_i\vee v_{i+1})\\&=E^+(e_i,z)E^-(e_i,z)v_i\vee E^+(-e_{i+1},z)E^-(-e_{i+1},z)v_{i+1}\end{aligned}$$

therefore $X_0(a_i)(v\otimes e^r)$ is determined by the coefficient of $z^{-2-2(a_i,r)}$ in ψ. Now, the degree of the lowest-degree term of z in $E^-(e_i,z)v_i$ is $2\deg(v_i)$; the degree of the lowest-degree term of z in $E^-(-e_{i+1},z)$ is $2\deg(v_{i+1})$, hence $X_0(a_i)(v\otimes e^r)=0$ if and only if

$$(a_i,r)\geqslant-\deg(v_i)-\deg(v_{i+1})\qquad 1\leqslant i<l.$$

When $a=a_l\in\Delta_S$, then

$$X(a_l,z)(v\otimes e^r)=F^+(a_l,z)F^-(a_l,z)E^+(a_l,z)E^-(a_l,z)v\otimes z^{1+2(a_l,r)}\varepsilon(a_l,r)\,e^{r+a_l}.$$

Denote

$$\psi=F^+(a_l,z)F^-(a_l,z)E^+(a_l,z)E^-(a_l,z)v_l$$

therefore $X_0(a_l)(v\otimes e^r)$ is determined by the coefficient of $Z^{-1-2(a_l,r)}$ in ψ. Now, the degree of the lowest-degree term of z in $E^-(a_l,z)v_l$ is $2\deg(v_l)$; the degree of the lowest-degree term of z in $F^-(a_l,z)\tilde{v}_l$ is $2\deg(\tilde{v}_l)$, hence $X_0(a_l)(v\otimes e^r)=0$ if and only if

$$(a_l,r)\geqslant-\deg v_l-\deg\tilde{v}_l.$$

Now $X_1(-\theta)(v\otimes e^r)=0$, but

$$X(-\theta,z)(v\otimes e^r)=E^+(-\theta,z)E^-(-\theta,z)v\otimes z^{2-2(\theta,r)}\varepsilon(-\theta,r)\,e^{r-\theta}$$

denote $\psi=E^+(-e_1,z)E^-(-e_1,z)\vee E^+(-e_2,z)E^-(-e_2,z)v_2$, then we can prove that $X_1(-\theta)(v\otimes e^r)=0$, if and only if

$$-(\theta,r)\geqslant-\deg v_l-\deg v_2-1$$

in the same way.

So $(a_i,r)\geqslant0$, $1-(\theta,r)\geqslant0$ are always true, they imply $r=0$ or $r=e_1$. When $r=0$, we have $\deg(v_1)=\ldots\deg(v_l)=\deg(\tilde{v}_l)=0$, i.e. $v=1$; when $r=e_1$, we have $\deg(v_2)=\ldots\deg(v_l)=\deg(\tilde{v}_l)=0$, and $1\geqslant-\deg(v_1)-\deg(v_2)$, but $(\theta,e_1)=1$, so we prove that $\deg(v_1)=0$, that is $v=1$. □

Theorem 2. Let V_0 be the subspace with a basis $v\otimes e^r$, where $r\in L$ and $\deg(v\otimes e^r)\in\mathbb{Z}^-$ and $V_{1/2}$ be the subspace with a basis $v\otimes e^r$, where $r\in L$ and $\deg(v\otimes e^r)\in\mathbb{Z}^- -\frac{1}{2}$ in the representation space V of the integrable representation (V,ρ), then we have a direct sum of subspaces $V=V_0\oplus V_{\frac{1}{2}}$, where V_0 and $V_{1/2}$ are two minimal invariant subspaces of vertex representation (V,ρ).

Proof. By lemma 4.2, we know that both V_0 and $V_{1/2}$ are the direct sums of irreducible highest-weight modules. But V_0 has the highest weight a_{-1} and $V_{1/2}$ has the highest weight $a_{-1}+e_1-\frac{1}{2}\delta$, therefore V_0 and $V_{1/2}$ are both integrable highest-weight modules and they are irreducible as well. □

5. Vertex representation of $G_2^{(1)}$

Now we consider the finite-dimensional simple Lie algebra G_2 over C. Let H be its Cartan subalgebra, H* is the dual space of H, then there exists an inner product (a, b) and a normal orthogonal basis $\{e_1, e_2\}$ in H* such that $\Delta=\Delta_S\cup\Delta_L$ and the simple root system

$$\pi=\left\{a_1=\sqrt{2}e_1,\ a_2=-\frac{1}{\sqrt{2}}e_1-\frac{1}{\sqrt{6}}e_2\right\} \tag{5.1}$$

$$\Delta_L=\left\{\pm a_1=\pm\sqrt{2}e_1,\ \pm(a_1+3a_2)=\mp\left(\frac{1}{\sqrt{2}}e_1+\frac{\sqrt{3}}{2}e_2\right)\right.$$
$$\left.\pm(2a_1+3a_2)=\pm\left(\frac{1}{\sqrt{2}}e_1-\sqrt{\frac{2}{3}}e_2\right)\right\} \tag{5.2}$$

$$\Delta_S=\left\{\pm a_2=\mp\left(\frac{1}{\sqrt{2}}e_1+\frac{1}{\sqrt{6}}e_2\right),\ \pm(a_1+a_2)=\pm\left(\frac{1}{\sqrt{2}}e_1-\frac{1}{\sqrt{6}}e_2\right)\right.$$
$$\left.\pm(a_1+2a_2)=\mp\sqrt{\frac{2}{3}}e_2\right\}. \tag{5.3}$$

Suppose that $\{e_1^\vee, e_2^\vee\}$ is the dual basis of $\{e_1, e_2\}$ in H, then the simple root system of G_2^* is $\pi^\vee=\{a_1^\vee=\sqrt{2}e_1^\vee,\ a_2^\vee=-(1/\sqrt{2})e_1^\vee-\sqrt{3/2}\,e_2^\vee\}$. Let $\gamma: e_i^\vee\to e_i$, $i=1,2$ be the natural linear isomorphism of H to H*, then we have $\gamma(a_1^\vee)=a_1$, $\gamma(a_2^\vee)=3a_2$, and

$$(a,b)=b(\gamma^{-1}(a))\quad (x,y)=\gamma(x)(y)\qquad \forall a,b\in H^*\quad x,y\in H. \tag{5.4}$$

Lemma 5.1. Set

$$L=\left\{m_1a_1-m_2a_2=\frac{2m_1+m_2}{\sqrt{2}}e_1+\frac{m_2}{\sqrt{2}}e_2\,\middle|\, m_1, m_2\in Z\right\}. \tag{5.5}$$

which is called root lattice. Let $p:\Delta_S\to L$ be a mapping such that

(i) $p(a)=p(a+b)$, $\forall a\in\Delta_S$, $b\in\Delta_L$, $a+b\in\Delta_S$;

(ii) $p(a)+p(b)=0$, $\forall a, b\in\Delta_S$, $a+b\in\Delta_L$;

(iii) $(a,-a)+2(p(a),p(-a))$ is an even integer number, $\forall a\in\Delta_S$;

(iv) $(a,b)+2(p(a),p(b))$ is an odd integer number, $\forall a, b\in\Delta_S$, $a+b\in\Delta$;

(v) $(-a,b)\geqslant\max(p(a),p(b)), -2(p(a),p(b))$, $\forall a, b\in\Delta$, $0\neq a+b\,\bar{\in}\,\Delta$,

then there exists an element $e\in\Delta_S$ such that

$$p(\pm a_2)=p(\pm(a_1+a_2))=p(\mp(a_1+2a_2))=\pm e. \tag{5.6}$$

Proof. Since

$$(\pm a_2)+(\pm a_1)=\pm(a_1+a_2)$$
$$(\pm a_2)+(\mp(a_1+3a_2))=\mp(a_1+2a_2).$$

By (i), we know that

$$p(\pm a_2)=p(\pm(a_1+a_2))=p(\mp(a_1+2a_2))=e_\pm.$$

By (ii) and $a_2+(-a_1-a_2)=-a_1\in\Delta_L$, we know that $e_++e_-=0$.

Let $e_+=e$, by (v), we have $0\leqslant(e,e)\leqslant\frac{2}{3}$, by (iii), we have that $-\frac{2}{3}-2(e,e)$ is equal to an even number, therefore $(e,e)+\frac{1}{3}$ is an integer number. Therefore $(e,e)=\frac{2}{3}$, i.e. $e\in\Delta_S$. □

Lemma 5.2. Suppose that $\varepsilon: L\times L\to\{\pm1\}$ is a mapping such that

$$\varepsilon(m_1a_1+m_2a_2, n_1a_1+n_2a_2)$$
$$=\varepsilon(a_1,a_1)^{m_1n_1}\varepsilon(a_1,a_2)^{m_1n_2}\varepsilon(a_2,a_1)^{m_2n_1}\varepsilon(a_2,a_2)^{m_2n_2} \tag{5.7}$$

where

$$\varepsilon(a_1,a_1)=\varepsilon(a_2,a_2)=\varepsilon(a_1,a_2)=1 \qquad \varepsilon(a_2,a_1)=-1 \tag{5.8}$$

then we have

(i) $\varepsilon(a,b)\varepsilon(a+b,c)=\varepsilon(b,c)\varepsilon(a,b+c), \forall a,b,c\in L$;
(ii) $\varepsilon(a,-a)=1, \forall a\in\Delta, a\neq\pm(a_1+a_2), a\neq\pm(a_1+3a_2)$;
(iii) $\varepsilon(a,-a)=1, a=\pm(a_1+a_2),\pm(a_1+3a_2)$;
(iv) $\varepsilon(a,b)\varepsilon(b,a)=(-1)^{(a,b)}, \forall a\in\Delta_L, b\in\Delta$;
(v) $\varepsilon(a,b)\varepsilon(b,a)=(-1)^{(a,b)+2(p(a),p(b))}, \forall a,b\in\Delta_S$;
(vi) $\varepsilon(a,b)=-\varepsilon(b,a), \forall a,b,a+b\in\Delta$.

Proof. By direct computation, we can prove that (i)-(v) are true. Since $(a,b)=-1, \forall a, a+b\in\Delta, b\in\Delta_L$; $(a,b)=\frac{1}{3}, \forall a,b\in\Delta_S, a+b\in\Delta_L$; $(a,b)=-\frac{1}{3}, a,b,a+b\in\Delta_S$, thus (vi) holds by (iv) and (v). □

We know that $G_2^{(1)}$ has a basis $c, d, t^n\otimes a_i^\vee, i=1,2, t^n\otimes e_\alpha, \alpha\in\Delta$, and the multiplication table is

$$\begin{aligned}
&[c,d]=0 \qquad [v,t^n\otimes x]=0 \qquad [d,t^n\otimes x]=n(t^n\otimes x) \qquad \forall x\in G_2\\
&[t^n\otimes\gamma^{-1}(a), t^m\otimes\gamma^{-1}(b)]=n\delta_{n,-m}(a,b)c \qquad \forall a,b\in H^*\\
&[t^n\otimes\gamma^{-1}(a), t^m\otimes e_b]=(a,b)t^{n+m}\otimes e_b \qquad \forall a,b\in H^*\\
&[t^n\otimes e_a, t^m\otimes e_{-a}]=\frac{2}{(a,a)}\varepsilon(a,-a)(t^{n+m}\otimes\gamma^{-1}(a)+n\delta_{m,-m}c) \qquad \forall a\in\Delta\\
&[t^n\otimes e_a, t^m\otimes e_b]=\frac{5+3(a,b)}{2}\varepsilon(a,b)e_{a+b} \qquad \forall a,b,a+b\in\Delta\\
&[t^n\otimes e_a, t^m\otimes e_b]=0 \qquad \forall a,b\in\Delta, a+b\neq0, a+b\bar{\in}\Delta
\end{aligned} \tag{5.9}$$

where $n,m\in\mathbb{Z}$, $\delta_{j,-k}$ is the Kronecker symbol, and $\varepsilon(a,b)$ is given by lemma 5.2.

In this section, the symbols are the same as in sections 1 and 2. Let

$$e=a_0=\sqrt{\tfrac{2}{3}}e_2 \tag{5.10}$$

in (5.6), and

$$p(\pm a_1)=p(\pm(a_1+3a_2))=p(\pm(2a_1+3a_2))=0 \tag{5.11}$$

and let S^- have a basis

$$1 \qquad h_1(n_1)=t^{n_1}\otimes\gamma^{-1}(e_1) \qquad h_2(n_2)=t^{n_2}\otimes\gamma^{-1}(e_2) \tag{5.12}$$

where $n_1 \in \mathbb{Z}^- - \{0\}$, $n_2 \in \frac{1}{3}\mathbb{Z}^- - \{0\}$. Then $V = S(S^-) \otimes C\{L\}$ and denote its complete space by $\tilde{V}$, thus we have c_0, d_0, ε_a, $H_i(n_i)$, $i = 1, 2$, $n_1 \in \mathbb{Z}$, $n_2 \in \frac{1}{3}\mathbb{Z}$,

$$\tilde{X}(a, z)(v \otimes e^r) = z^{(3/2)(a,a)+3(a,r)}(v \otimes e^{a+r}) \tag{5.13}$$

$$E^{\pm}(a, z) = \exp\left(\pm \sum_{n=1}^{\infty} \frac{1}{n} z^{\pm 3n} a(\mp n)\right) \tag{5.14}$$

$$F^{\pm}(a, z) = \exp\left(\pm \sum_{n=1}^{\infty} \frac{3}{3n-1} z^{\pm(3n-1)} a\left(\mp \frac{3n-1}{3}\right)\right) \times \exp\left(\pm \sum_{n=1}^{\infty} \frac{3}{3n-2} z^{\pm(3n-2)} a\left(\mp \frac{3n-2}{3}\right)\right) \tag{5.15}$$

where z is a complex variable. We can prove that

$$E^-(a, z)E^+(b, w) = z^{-3(a,b)}(z^3 - w^3)^{(a,b)} E^+(b, w)E^-(a, z) \qquad |w| < |z| \tag{5.16}$$

$$F^-(a, z)F^+(b, w) = (z^3 - w^3)^{-(p(a),p(b))}(z - w)^{3(p(a),p(b))} F^+(b, w)F^-(a, z) \qquad |w| < |z|. \tag{5.17}$$

Let

$$X(a, z) = F^+(p(a), z)F^-(p(a), z)E^+(a, z)E^-(a, z)\tilde{X}(a, z)\varepsilon_a \qquad \forall a \in \Delta \tag{5.18}$$

so, the Laurent series $X(a, z) = \sum_{n=-\infty}^{\infty} X_{n/3}(a) z^{-n}$ introduces the vertex operators $X_n(a)$, $\forall n \in \mathbb{Z}$, $a \in \Delta$. Let $w = \frac{1}{2}(-1 + \sqrt{-3})$, then

$$X(a, z)X(b, z_0) = \varepsilon(a, b)(zz_0^{-1})^{(3/2)(a,a)+3(a,r)} z_0^{-3(a,b)}(z - z_0)^{(a,b)+2(p(a),p(b))} \times (z - wz_0)^{(a,b)-(p(a),p(b))}(z - w^2 z_0)^{(a,b)-(p(a),p(b))} Y(a, b; z, z_0) \tag{5.19}$$

where

$$Y(a, b; z, z_0) = F^+(p(a), z)F^+(p(b), z_0)F^-(p(a), z)F^-(p(b), z_0)E^+(a, z)E^-(b, z_0) \times E^+(a, z)E^-(b, z_0)\tilde{X}(a + b, z_0)\varepsilon_{a+b}. \tag{5.20}$$

Obviously, the Laurent series of $X(a, z)$, $X(a, wz)$, $X(z, w^2 z)$ give the same vertex operators $X_n(a)$, $\forall n \in \mathbb{Z}$, then

$$Y(a, b; w^i z_0, z_0) = X(a + b, z_0) \qquad i = 0, 1, 2 \tag{5.21}$$

when $a \in \Delta_L$;

$$Y(a, b; w^i z_0, z_0)(v \otimes e^r) = X(a + b, w^{2i} z_0) w^{i+3i(a+b,r)}(v \otimes e^r) \qquad i = 1, 2 \tag{5.22}$$

when $a, b \in \Delta_S$, $a + b \in \Delta_S$, $p(a) = p(b) = -p(a + b)$. We know that

$$\varepsilon(a, b) = \varepsilon(b, a)(-1)^{(a,b)+2(p(a),p(b))} \qquad \forall a, b \in \Delta \tag{5.23}$$

by lemma 5.2. As in section 3, we have

$$[X_n(a), X(b, z_0)](v \otimes e^r) = \frac{\varepsilon(a, b)}{2\pi\sqrt{-1}} \int_D f_r(a, b; z, z_0)\, \mathrm{d}z \tag{5.24}$$

where $D=\{0<r_0<|z|<r_1\}\ni z_0$ and

$$f_r(a.b;\, z.z_0)=z^{3n-1}\varepsilon(a,b)X(a,z)X(b,z_0)(v\otimes e^r). \tag{5.25}$$

For further computation, we consider the singularity of function f_r in the following:

(i) when $a, b\in\Delta$ and $a+b\neq 0$, $a+b\bar{\in}\Delta$, then f is not a singularity in D;

(ii) when $a\in\Delta_L$, b, $a+b\in\Delta$ and $(a,b)=-1$, $p(a)=0$, then the singularity appears in $(z-z_0)^{-1}(z-wz_0)^{-1}(z-w^2z_0)^{-1}$;

(iii) when $a, b, a+b\in\Delta_S$ and $(a,b)=-\frac{1}{3}$, $p(a)=p(b)=-p(a+b)$, then the singularity appears in $(z-z_0)(z-wz_0)^{-1}(z-w^2z_0)^{-1}$;

(iv) when $a, b\in\Delta_S$, $a+b\in\Delta_L$, $(a,b)=\frac{1}{3}$ and $p(a)=-p(b)$, then the singularity appears in $(z-z_0)^{-1}(z-wz_0)(z-w^2z_0)$;

(v) when $b=-a\in\Delta_L$, $(a,b)=-2$ and $p(a)=0$, then the singularity appears in $(z-z_0)^{-2}(z-wz_0)^{-2}(z-w^2z_0)^{-2}$;

(vi) when $b=-a\in\Delta_S$, $(a,b)=-\frac{2}{3}$ and $p(b)=-p(a)$, then the singularity appears in $(z-z_0)^{-1}(z-wz_0)^0(z-w^2z_0)^0$.

Then we have the following immediately:

(i) $[X_n(a), X_m(b)]=0$, $\forall n, m\in\mathbb{Z}$, $a, b\in\Delta$, $0\neq a+b\bar{\in}\Delta$.

(ii) $[X_n(a), X(b,z_0)](v\otimes e^r)$

$$\begin{aligned}&=\varepsilon(a,b)z_0^{3n}[(1-w)^{-1}(1-w^2)^{-1}+w^{3n+2+3(a,r)}(w-1)^{-1}(w-w^2)^{-1}\\&\quad+w^{6n+4+6(a,r)}(w^2-1)^{-1}(w^2-w)^{-1}]X(a+b,z_0)(v\otimes e^r)\\&=\varepsilon(a,b)z_0^{3n}X(a+b,z_0)(v\otimes e^r),\end{aligned}$$

since $(a,r)\in\mathbb{Z}$, $a=\pm a_1, \pm(a_1+3a_2), \pm(2a_1+3a_2)$. Therefore we have proved that

$$[X_n(a), X_m(b)]=\varepsilon(a,b)X_{n+m}(a+b) \qquad \forall a\in\Delta_L,\ b,\ a+b\in\Delta.$$

(iii)

$[X_n(a), X(b,z_0)](v\otimes e^r)$

$$=\varepsilon(a,b)[-z_0^{3n}w^{3(2a+b,r)}X(a+b,w^2z_0)-z_0^{3n}w^{6(2a+b,r)}X(a+b,wz_0)](v\otimes e^r)$$

since

$$a,\ b,\ a+b\in\Delta_S,$$

then

$$a, b\in\{a_2, a_1+a_2, -a_1-2a_2\}$$

or

$$a, b\in\{-a_2, -a_1-a_2, a_1+2a_2\},$$

thus $2a+b=pa_1+qa_2$, here $3|q$, therefore $(2a+b,r)\in\mathbb{Z}$, $\forall r\in L$. In this case, $w^{3(2a+b,r)}=1$, hence

$$[X_n(a), X_m(b)]=-2\varepsilon(a,b)X_{n+m}(a+b) \qquad \forall a,\ b,\ a+b\in\Delta_S,$$

(iv) $[X_n(a), X(b,z_0)](v\otimes e^r)=\varepsilon(a,b)z_0^{3n}(1-w)(1-w^2)X(a+b,z_0)(v\otimes e^r)$
that is

$$[X_n(a), X_m(b)]=3\varepsilon(a,b)X_{n+m}(a+b) \qquad \forall a,\ b\in\Delta_S,\ b,\ a+b\in\Delta_L.$$

(v) Since

$b=-a\in\Delta_L$, $X(a,z)=E^{+}(a,z)E^{-}(a,z)\tilde{X}(a,z)\varepsilon_a$, $X(-a,z_0)$

$$=E^{+}(-a,z_0)E^{-}(-a,z_0)\tilde{X}(-a,z_0)\varepsilon_{-a}.$$

Put $z^3=x$, $z_0^3=x_0$, we have

$$[X_n(a),X_m(-a)]=\frac{2}{(a,a)}\varepsilon(a,-a)[a(n+m)+n\delta_{n,-m}c_0]$$

as section 3.

(vi) Since $b=-a\in\Delta_S$ and $Y(a,-a;z_0,z_0)=c_0$, we have

$[X_n(a),X(-a,z_0)](v\otimes e^r)$

$$=\varepsilon(a,-a)\Bigg[(3n+3(a,r))z_0^{3n}Y(a,-a;z_0,z_0)+z_0^{3n+1}$$

$$\times\Bigg(3\sum_{\substack{m\neq 0(\mathrm{mod}\ 3)\\ m\in\mathbb{Z}}} z_0^{m-1}p(a)\left(-\frac{m}{3}\right)$$

$$+3\sum_{m\in\mathbb{Z}} z_0^{3m-1}a(-m)-3z_0^{-1}a(0)\Bigg)\Bigg](v\otimes e^r)$$

therefore

$[X_n(a),X(-a,z_0)]$

$$=3\varepsilon(a,-a)\Bigg[nz_0^{3n}c_0+\sum_{\substack{m\neq 0(\mathrm{mod}\ 3)\\ m\in\mathbb{Z}}} z_0^{3n+m}p(a)\left(-\frac{m}{3}\right)+\sum_{m\in\mathbb{Z}} z_0^{3n+3m}a(-m)\Bigg],$$

hence

$$[X_n(a),X_m(-a)]=3\varepsilon(a,-a)[a(n+m)+n\delta_{n,-m}c_0]\qquad \forall a\in\Delta_S.$$

So we have proved, the following theorem.

Theorem 3. The mapping $\rho: c\to c_0, d\to d_0, t^n\otimes\gamma^{-1}(e_i)\to H_i(n), i=1,2, t^n\otimes e_a\to X_n(a), \forall a\in\Delta_L;\ t^n\otimes e_a\to -X_n(a), \forall a\in\Delta_S, n\in\mathbb{Z}$ has given a linear representation from $G_2^{(1)}$ to the operator set on V, denoted by (V,ρ). This representation is called vertex representation of $G_2^{(1)}$.

Since we have six different choices of e in lemma 5.1, similarly, we can get six vertex representation for $G_2^{(1)}$.

6. The decomposition of vertex representations of $G_2^{(1)}$

Let H_1 be the Cartan subalgebra of $G_2^{(1)}$, H_1^* is the dual space of H_1, then there exist a basis in H_1:

$$a_{-1}^\vee=d\qquad a_0^\vee=c-2a_1^\vee-a_2^\vee\qquad a_1^\vee\qquad a_2^\vee\tag{6.1}$$

and a basis in $H_1^*: a_{-1}, a_0, a_1, a_2$ such that a_0, a_1, a_2 form the simple root system. Then we have a non-degenerate symmetric bilinear function in H_1^*, such that e_1, e_2 form a part of its orthogonal basis, and also $(a_{-1},a_{-1})=0, (a_{-1},a_i)=0, i=1,2, (a_0,a_0)=2, (a_0,a_1)=-1, (a_0,a_2)=0, (a_0,a_{-1})=1.$

Suppose that $\gamma: H_1 \to H_1^*$ is a linear mapping, such that $\gamma(a_2^\vee) = 3a_2$, $\gamma(a_i^\vee) = a_i$, $-1 \leq i \leq 1$, then

$$a(x) = (a, \gamma(x)) \quad (a, b) = b(\gamma^{-1}(a))$$
$$(x, y) = \gamma(x)(y) \qquad \forall a, b \in H_1^* \quad x, y \in H_1. \tag{6.2}$$

The imaginary root system of $G_2^{(1)}$ is $\Delta_{im} = \{k\delta | k \neq 0, k \in Z\}$, where $\delta = a_0 + 2a_1 + 3a_2$, then $(\delta, a_{-1}) = 1$, $(\delta, a_0) = 0$, $(\delta, L) = 0$.

We know that $\rho(H_1)$ has a basis d_0, c_0, $H_i(0)$, $i = 1, 2$, so the root subspace decomposition is

$$V = \sum_{\lambda \in P(V)} V_\lambda \tag{6.3}$$

acted by the operator set $\rho(H_1)$, where $P(V) \subset H_1^*$ is the weight system of vertex representation (V, ρ) of $G_2^{(1)}$. Similarly, we have the following.

Lemma 6.1. The weight space V_λ has the basis $v \otimes e^r$, where $r \in L$ and $v = 1$ or $v = h_{i_1}(n_1) \vee \ldots \vee h_{i_t}(n_t)$. In this case $\lambda = a_{-1} + r + (\deg(v \otimes e^r))\delta$. And $\deg(v)$, r are uniquely determined by λ.

Lemma 6.2. For any $\lambda \in P(V)$, we have $\lambda \leq a_{-1}$ or $\lambda \leq a_{-1} - \sqrt{\tfrac{2}{3}}\, e_2 - \tfrac{1}{3}\delta$, or $\lambda \leq a_{-1} - \sqrt{\tfrac{2}{3}}\, e_2 - \tfrac{2}{3}\delta$.

Lemma 6.3. (V, ρ) is an integrable representation. The proofs of these lemmas are similar to the proofs of lemma 4.1, 4.2, 4.3, respectively.

Lemma 6.4. $P_+ \cap P(V) = \{a_{-1} + (\deg v)\delta,\ a_{-1} - \sqrt{\tfrac{2}{3}}\, e_2 + (\deg v - \tfrac{1}{3})\delta\}$.

Proof. For any $\lambda \in P(V)$, then $\lambda = a_{-1} + r + (\deg(v \otimes e^r))\delta$, by lemma 6.1. But $\lambda \in P_+$, so $\lambda(a_i^\vee) \in Z^+, 0 \leq i \leq 2$. Let $r = m_1 a_1 + m_2 a_2$, $m_1, m_2 \in \mathbb{Z}$, since $(\lambda, a_1) = (r, a_1) = 2m_1 - m_2 \in \mathbb{Z}^+$, $\lambda(a_2^\vee) = 3(\lambda, a_2) = -3m_1 + 2m_2 \in \mathbb{Z}^+$, $(\lambda, a_0) = (a_{-1} + r, a_0) = 1 - m_1 \in \mathbb{Z}^+$. Hence we have proved $m_1 = m_2 = 0$ or $m_1 = 1, m_2 = 2$, i.e. $r = 0$ or $r = a_1 + 2a_2 = -\sqrt{\tfrac{2}{3}}\, e_2$. □

Lemma 6.5. Only $1 \otimes e^0$, $1 \otimes e^{-\sqrt{2/3}e_2}$ and $h_2(-\tfrac{1}{3}) \otimes e^{-\sqrt{2/3}e_2}$ are the highest-weight vectors in the representation space V, and the highest weights are a_{-1}, $a_{-1} - \sqrt{\tfrac{2}{3}}\, e_2 - \tfrac{1}{2}\delta$, $a_{-1} - \sqrt{\tfrac{2}{3}}\, e_2 - \tfrac{2}{3}\delta$, respectively.

Proof. We know that the generators are $\rho(t \otimes e_{-\theta}) = X_1(-\theta)$, $\rho(t^{-1} \otimes e_\theta) = X_{-1}(\theta)$, $\rho(1 \otimes e_{a_i}) = X_0(a_i)$, $\rho(1 \otimes e_{-a_i}) = X_0(-a_i)$, $i = 1, 2$.

Let $v = 1$ or $v = v_1 \vee v_2 \vee \tilde{v}_2$, where $v_i = h_i(n_{i1}) \vee \ldots \vee h_i(n_{is_i})$, $i = 1, 2$, $n_{ij} \in \mathbb{Z}^- - \{0\}$, and

$$\tilde{v}_2 = h_2(-\tfrac{1}{3}(3m_1 + 1)) \vee \ldots \vee h_2(-\tfrac{1}{3}(3m_5 + 1)) \vee h_2(-\tfrac{1}{3}(3n_1 + 2)) \vee \ldots \vee h_2(-\tfrac{1}{3}(3n_t + 2)),$$
$$m_i, n_j \in \mathbb{Z}^-.$$

Since $\tilde{X}(a, z)\varepsilon_a(v \otimes e^r) = \varepsilon(a, r) z^{(3/2)(a,a) + 3(a,r)}(v \otimes e^{r+a})$. When $a = a_1 \in \Delta_L$, then

$$X(a, z)(v \otimes e^r) = E^+(a_1, z)E^-(a_1, z)v \otimes z^{3+3(a_1, r)}\varepsilon(a_1, r)\, e^{a_1 + r}.$$

Denote

$$\psi_1 = E^+(a_1, z)E^-(a_1, z)v_1 = E^+(\sqrt{2}\, e_1, z)E^-(\sqrt{2}\, e_1, z)v_1$$

therefore $X_0(a_1)(v\otimes e^r)$ is determined by the coefficient of $z^{-3-3(a,r)}$ in ψ_1. Now, the degree of the lowest-degree term of z in $E^-(\sqrt{2}\,e_1, z)$ is $3\deg(v_1)$, hence $X_0(a_1)(v\otimes e^r)=0$ if and only if $(a_1, r)\geq -\deg v_1$. When $a=a_2\in\Delta_S$, then

$$X(a_2, z)(v\otimes e^r) = F^+(\sqrt{\tfrac{2}{3}}\,e_2, z)F^-(\sqrt{\tfrac{2}{3}}\,e_2, z)E^+(a_2, z)E^-(a_2, z)v \otimes z^{1+3(a_2,r)}\varepsilon(a_2, r)\,e^{r+a_2}.$$

Denote

$$\psi_2 = F^+(\sqrt{\tfrac{2}{3}}\,e_2, z)F^-(\sqrt{\tfrac{2}{3}}\,e_2, z)\tilde{v}_2 \vee E^+(-(1/\sqrt{2})\,e_1, z) \times E^-(-(1/\sqrt{2})\,e_1, z)v_1 \vee E^+(-(1/\sqrt{6})\,e_2, z)E^-(-(1/\sqrt{6})\,e_2, z)v_2$$

therefore $X_0(a_2)(v\otimes e^r)$ is determined by the coefficient of $z^{-1-3(a,r)}$ in ψ_2. Now, the degrees of the lowest-degree terms of z in $E^-(-(1/\sqrt{2})\,e_1, z)v_1$, $E^-(-(1/\sqrt{6})\,e_2, z)v_2$ and $F^-(\sqrt{\tfrac{2}{3}}\,e_2, z)\tilde{v}_2$ are $3\deg(v_1)$, $3\deg(v_2)$ and $3\deg(\tilde{v}_2)$, respectively, hence $X_0(a_2)(v\otimes e^r)=0$ if and only if $1+3(a_2, r)>-3\deg v_1-3\deg v_2-3\deg\tilde{v}_2$. For $\theta=2a_1+3a_2=(1/\sqrt{2})e_1-\sqrt{\tfrac{3}{2}}\,e_2\in\Delta_L$, we can prove $X_1(-\theta)(v\otimes e^r)=0$ if and only if $-(\theta, r)\geq -\deg v_1-\deg v_2-1$. By lemma 6.5, $r=0$ or $r=-\sqrt{\tfrac{2}{3}}\,e_2$. When $r=0$, then $\deg(v_1)=\deg(v_2)=\deg(\tilde{v}_2)=0$, i.e. $v=1$; when $r=-\sqrt{\tfrac{2}{3}}\,e_2$, then $\deg(v_1)=\deg(v_2)=0$ and $-\deg(\tilde{v}_2)<\tfrac{2}{3}$, therefore $\deg(\tilde{v}_2)=0$ or $\deg(\tilde{v}_2)=-\tfrac{1}{3}$, i.e. $v=1$ or $v=h_2(-\tfrac{1}{3})$. □

Theorem 4. Let $V_{j/3}$ be the subspace with a basis $v\otimes e^r$, where $r\in L$ and $\deg(v\otimes e^r)\in\mathbb{Z}^- -j/3, j=0,1,2$, then we have a direct sum of subspaces: $V=V_0+V_{1/3}+V_{2/3}$, where V_0, $V_{1/3}$, $V_{2/3}$, are three minimal invariant subspaces of vertex representation (V,ρ).

The proof is the same as that of theorem 2.

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References

[1] Frenkel and Kac 1980 Basic representations of affine Lie algebras and dual resonance models *Invent. Math.* **62** 23-66

[2] Segal G 1981 Unitary representations of some infinite dimensional groups *Commun. Math. Phys.* **80** 301-42

[3] Goddard P, Nahm W, Olive D and Schwimmer A 1986 Vertex operators for non-simple-laces algebras *Commun. Math. Phys.* **107** 179-212

[4] Lepowsky J and Prime M 1984 *Vertex Operators in Mathematics and Physics* ed J Lepowsky, S Mandelstem and I M Singer (*MSRI Publication* **3**) (Berlin: Springer) p 143

[5] Frenkel I B 1982 Representations of affine Lie algebras, Hecks modular forms and KdV type equations *Lie algebras and related topics* ed D Winter (*Lecture Notes in Math.* **933**) (Berlin: Springer) pp 71-110

[6] Kac V G 1983 *Infinite Dimensional Lie Algebras* (Basle: Birkhauser)